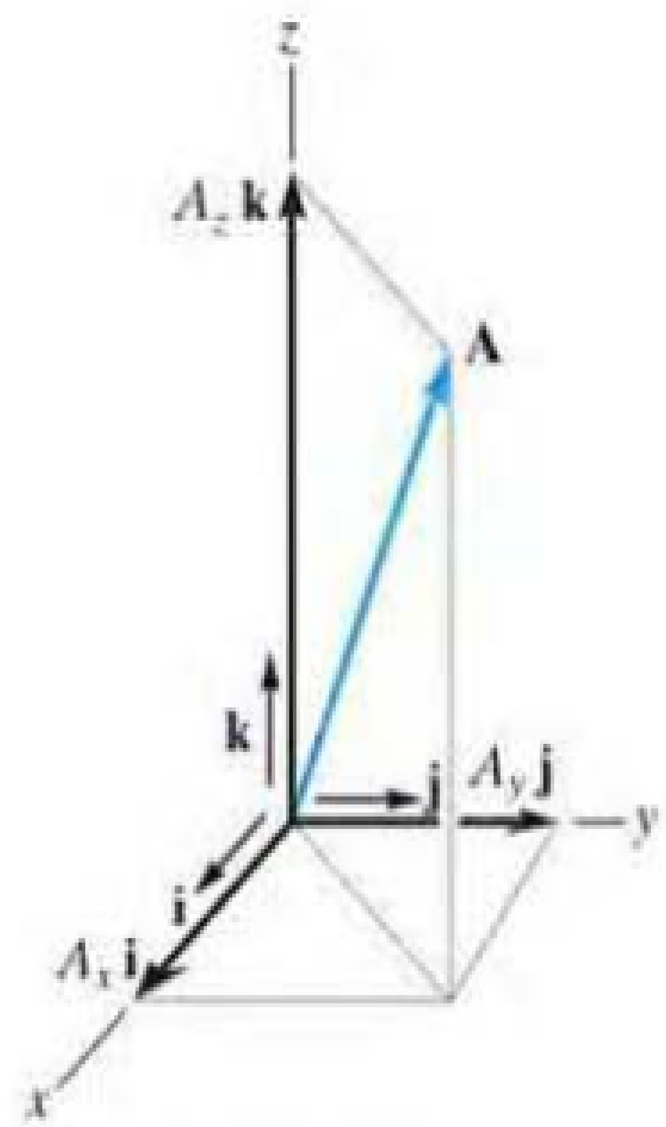


Fig. 2–24

Cartesian Vector Representation. Since the three components of **A** in Eq. 2–2 act in the positive **i**, **j**, and **k** directions, Fig. 2–24, we can write **A** in Cartesian vector form as

$$\boxed{\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}} \tag{2–3}$$

There is a distinct advantage to writing vectors in this manner. Separating the *magnitude* and *direction* of each *component vector* will simplify the operations of vector algebra, particularly in three dimensions.

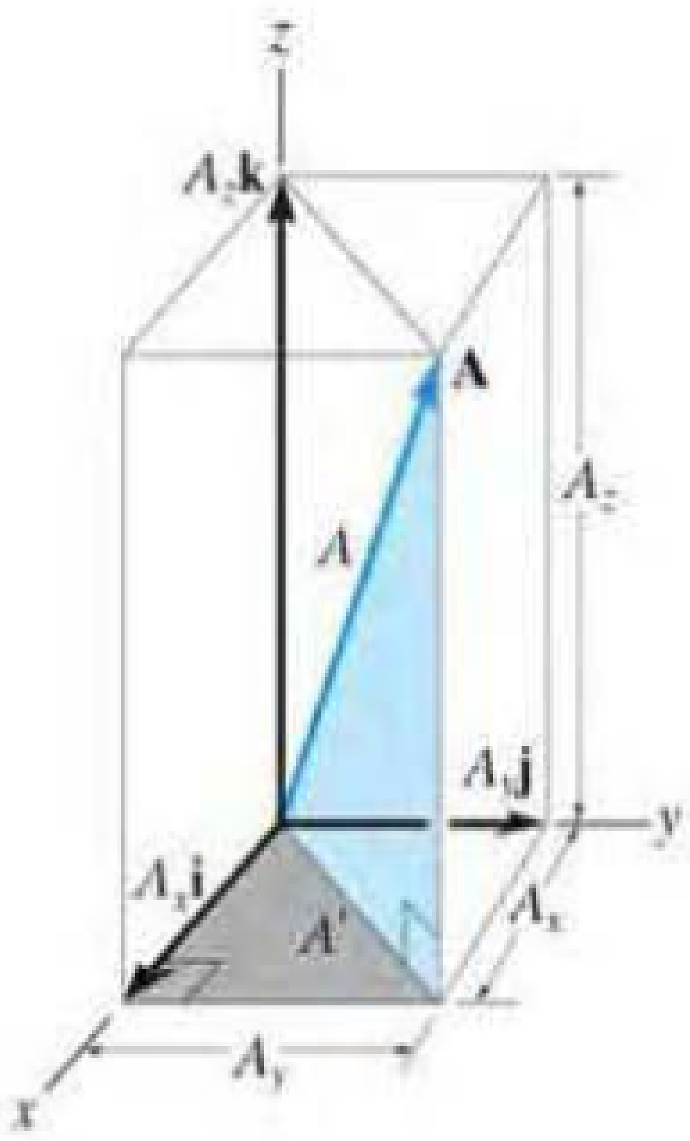


Fig. 2–25

Magnitude of a Cartesian Vector. It is always possible to obtain the magnitude of **A** provided it is expressed in Cartesian vector form. As shown in Fig. 2–25, from the blue right triangle, $A = \sqrt{A'^2 + A_z^2}$, and from the gray right triangle, $A' = \sqrt{A_x^2 + A_y^2}$. Combining these equations to eliminate A', yields

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{2–4}$$

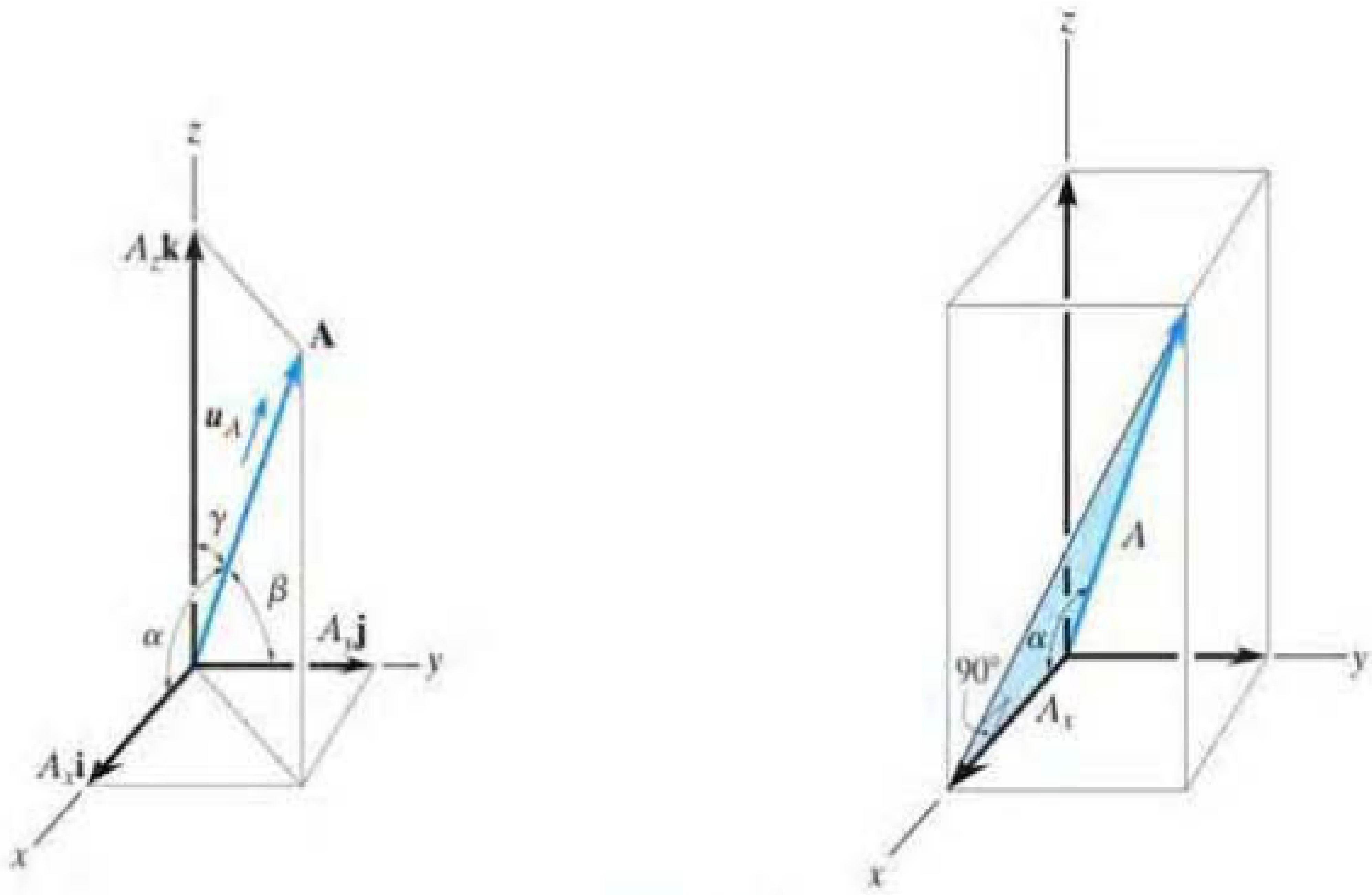
Hence, the magnitude of **A** *is equal to the positive square root of the sum of the squares of its components.*

Direction of a Cartesian Vector. We will define the *direction* of **A** by the *coordinate direction angles* α (alpha), β (beta), and γ (gamma), measured between the *tail* of **A** and the *positive* x, y, z axes provided they are located at the tail of **A**, Fig. 2–26. Note that regardless of where **A** is directed, each of these angles will be between 0° and 180°.

To determine α, β, and γ, consider the projection of **A** onto the x, y, z axes, Fig. 2–27. Referring to the blue colored right triangles shown in each figure, we have

$$\boxed{\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}} \tag{2–5}$$

These numbers are known as the *direction cosines* of **A**. Once they have been obtained, the coordinate direction angles α, β, γ can then be determined from the inverse cosines.



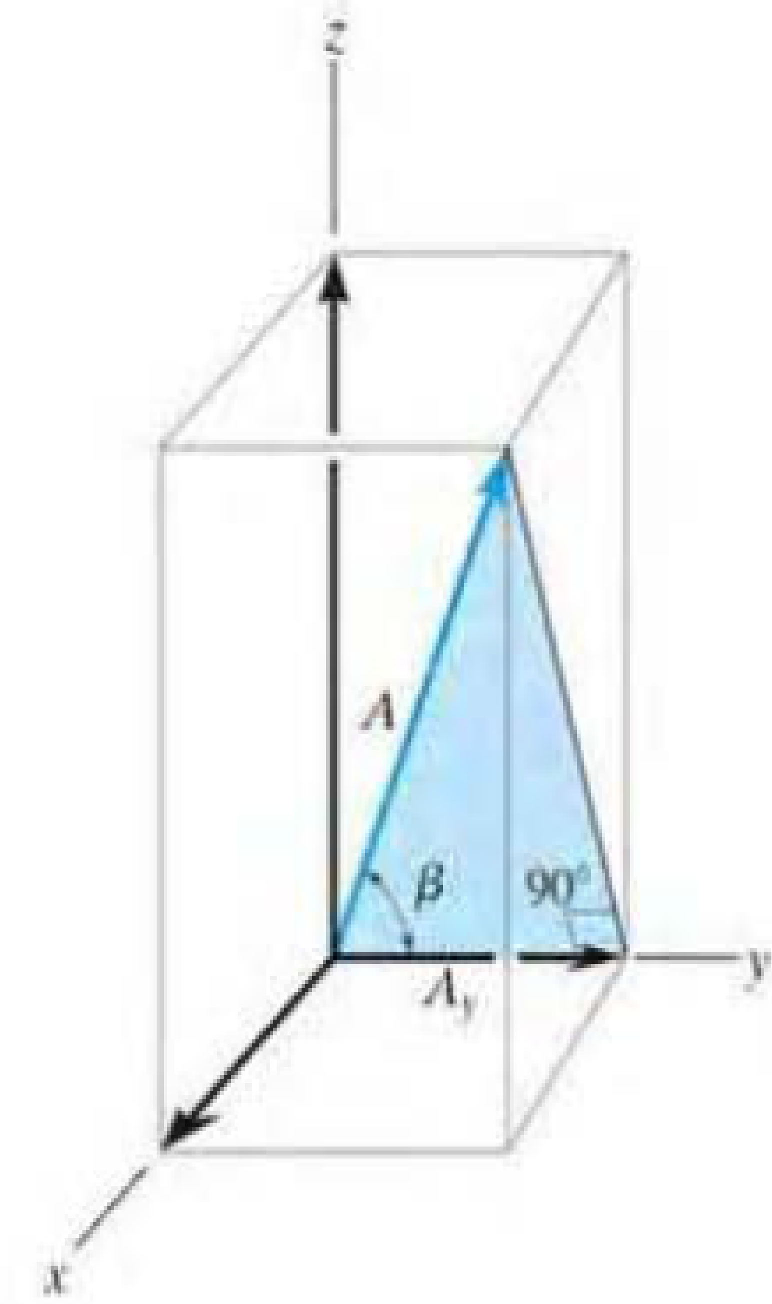


Fig. 2–26

An easy way of obtaining these direction cosines is to form a unit vector $\mathbf{u}_A$ in the direction of $\mathbf{A}$, Fig. 2–26. If $\mathbf{A}$ is expressed in Cartesian vector form, $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, then $\mathbf{u}_A$ will have a magnitude of one and be dimensionless provided $\mathbf{A}$ is divided by its magnitude, i.e.,

$$\boxed{\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}} \qquad (2\text{–}6)$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. By comparison with Eqs. 2–7, it is seen that *the* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *components of* $\mathbf{u}_A$ *represent the direction cosines of* $\mathbf{A}$, i.e.,

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \qquad (2\text{–}7)$$

Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\mathbf{u}_A$ has a magnitude of one, then from the above equation an important relation between the direction cosines can be formulated as

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1} \qquad (2\text{–}8)$$

Here we can see that if only *two* of the coordinate angles are known, the third angle can be found using this equation.

Finally, if the magnitude and coordinate direction angles of $\mathbf{A}$ are known, then $\mathbf{A}$ may be expressed in Cartesian vector form as

$$\boxed{\begin{aligned}\mathbf{A} &= A\mathbf{u}_A \\ &= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k} \\ &= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\end{aligned}} \qquad (2\text{–}9)$$

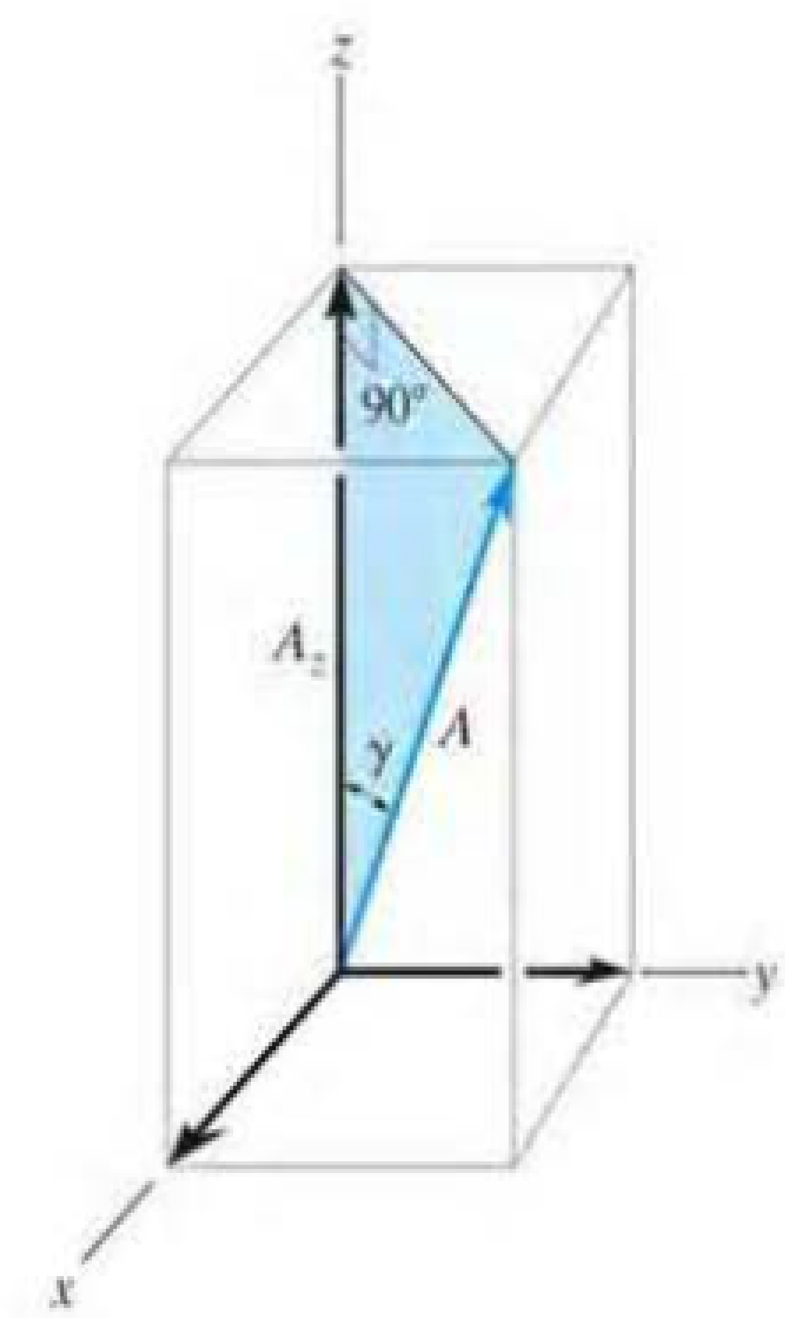


Fig 2–27

Addition of Cartesian Vectors

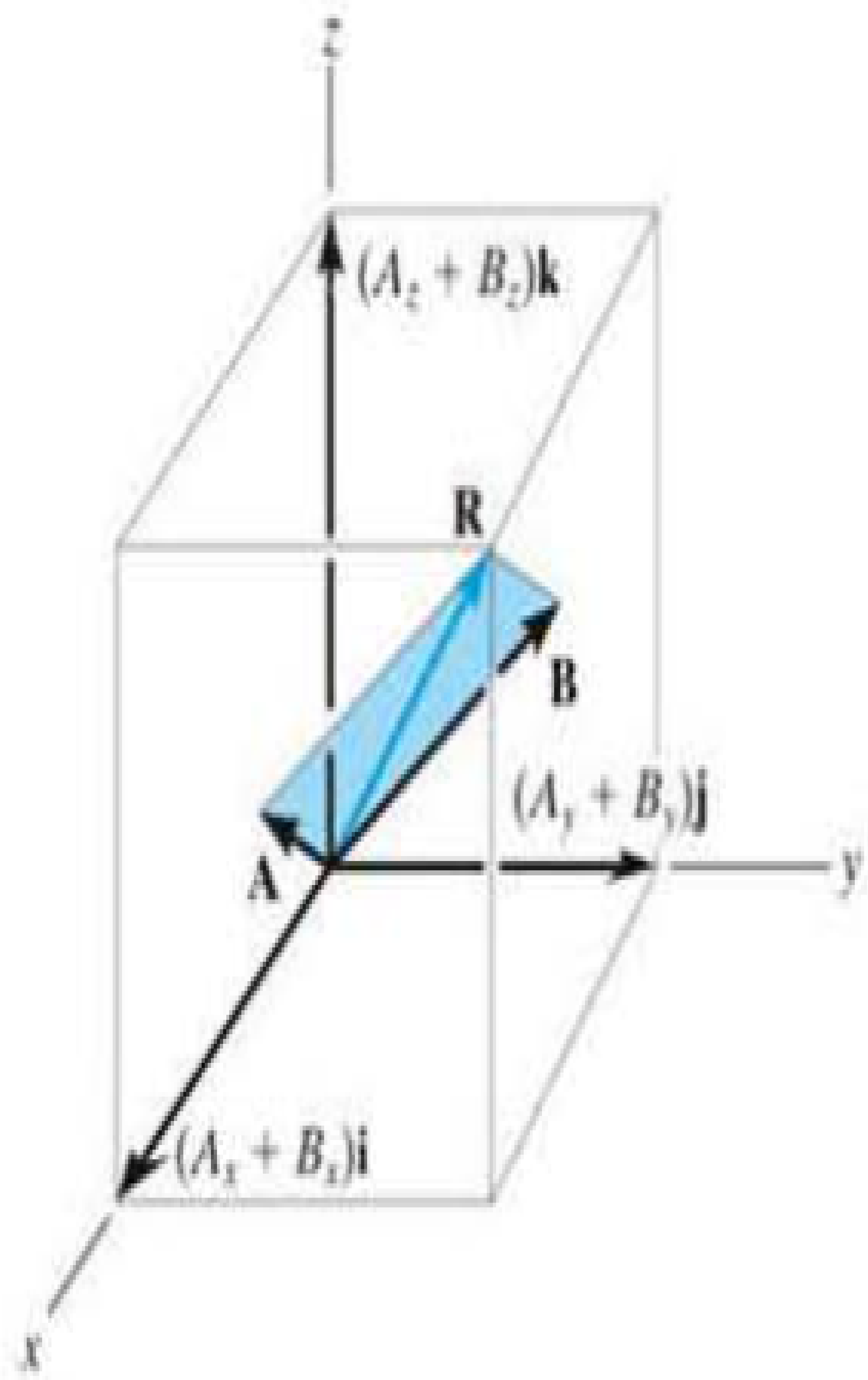


Fig. 2-29

The addition (or subtraction) of two or more vectors are greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, if $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, Fig. 2-29, then the resultant vector, $\mathbf{R}$, has components which are the scalar sums of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $\mathbf{A}$ and $\mathbf{B}$, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

If this is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

$$\boxed{\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}} \qquad (2\text{-}10)$$

Here ΣF_x, ΣF_y, and ΣF_z represent the algebraic sums of the respective x, y, z or $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of each force in the system.

Force Vector Directed Along a Line

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Fig. 2–38, where the force $\mathbf{F}$ is directed along the cord AB. We can formulate $\mathbf{F}$ as a Cartesian vector by realizing that it has the *same direction* and *sense* as the position vector $\mathbf{r}$ directed from point A to point B on the cord. This common direction is specified by the *unit vector* $\mathbf{u} = \mathbf{r}/r$. Hence,

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Although we have represented $\mathbf{F}$ symbolically in Fig. 2–38, note that it has *units of force*, unlike $\mathbf{r}$, which has units of length.

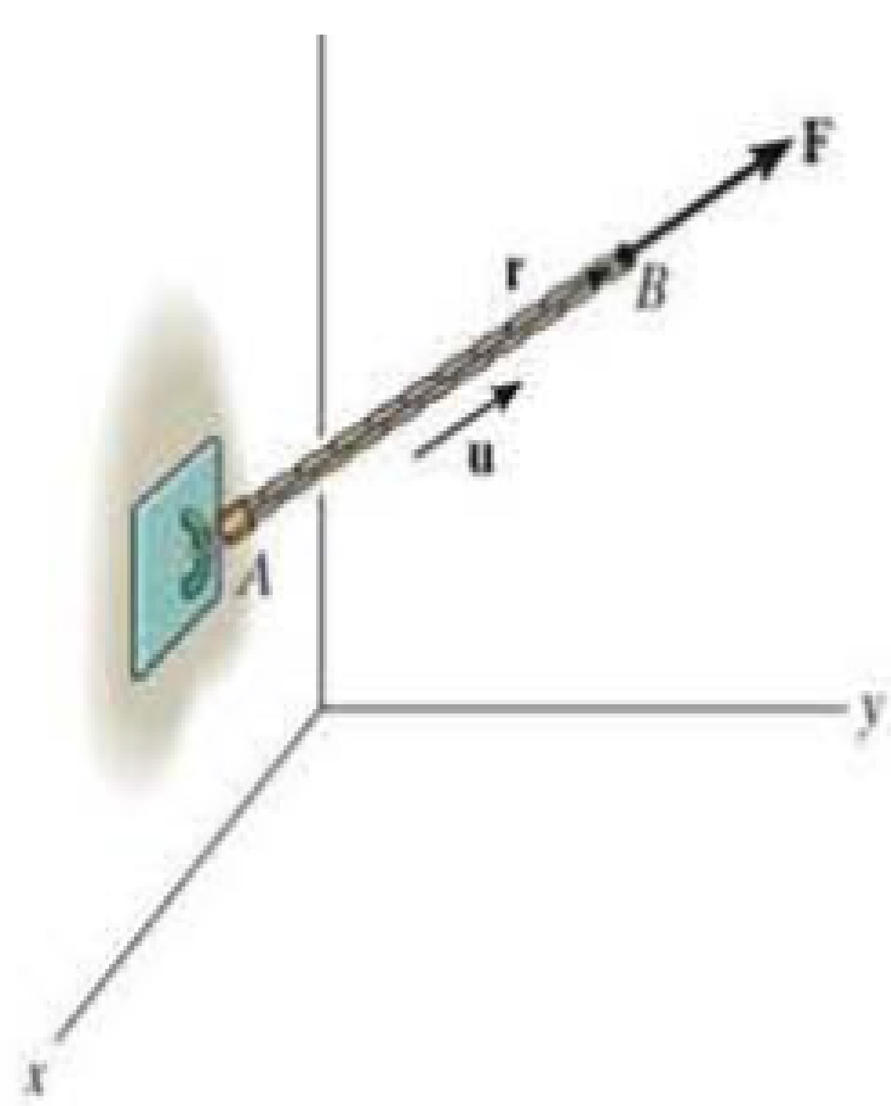


Fig. 2–38

Dot Product

Cartesian Vector Formulation. Equation 2–12 must be used to find the dot product for any two Cartesian unit vectors. For example, $\mathbf{i}\cdot\mathbf{i} = (1)(1)\cos 0^\circ = 1$ and $\mathbf{i}\cdot\mathbf{j} = (1)(1)\cos 90^\circ = 0$. If we want to find the dot product of two general vectors **A** and **B** that are expressed in Cartesian vector form, then we have

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_xB_x(\mathbf{i}\cdot\mathbf{i}) + A_xB_y(\mathbf{i}\cdot\mathbf{j}) + A_xB_z(\mathbf{i}\cdot\mathbf{k}) \\
&+ A_yB_x(\mathbf{j}\cdot\mathbf{i}) + (A_yB_y(\mathbf{j}\cdot\mathbf{j}) + A_yB_z(\mathbf{j}\cdot\mathbf{k}) \\
&+ A_zB_x(\mathbf{k}\cdot\mathbf{i}) + A_zB_y(\mathbf{k}\cdot\mathbf{j}) + A_zB_z(\mathbf{k}\cdot\mathbf{k})
\end{aligned}
$$

Carrying out the dot-product operations, the final result becomes

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z}$$

Applications.

Projection

$$proj\left(\vec{A}\big/\vec{B}\right) = \vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}$$

$$\overrightarrow{proj}\left(\vec{A}\big/\vec{B}\right) = \left(\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}\right)\frac{\vec{B}}{|\vec{B}|}$$

EXAMPLE

Express the force **F** shown in Fig. 2–30 as a Cartesian vector.

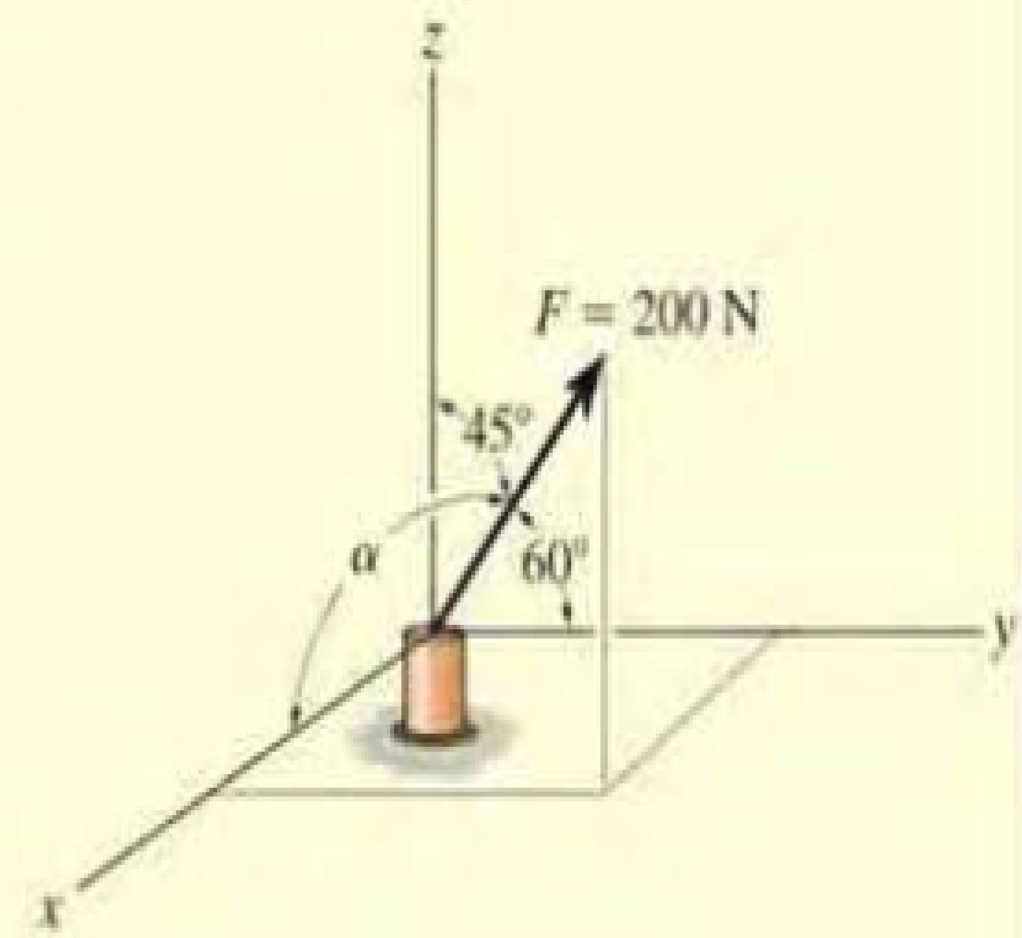


Fig. 2–30

SOLUTION

Since only two coordinate direction angles are specified, the third angle α must be determined from Eq. 2–8; i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60^\circ \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120^\circ$$

By inspection it is necessary that $\alpha = 60^\circ$, since $\mathbf{F}_x$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\mathbf{F} = F\cos\alpha\mathbf{i} + F\cos\beta\mathbf{j} + F\cos\gamma\mathbf{k}$$
$$= (200\cos 60^\circ \text{ N})\mathbf{i} + (200\cos 60^\circ \text{ N})\mathbf{j} + (200\cos 45^\circ \text{ N})\mathbf{k}$$
$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\} \text{ N}$$ *Ans.*

Show that indeed the magnitude of $F = 200$ N.

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–31*a*.

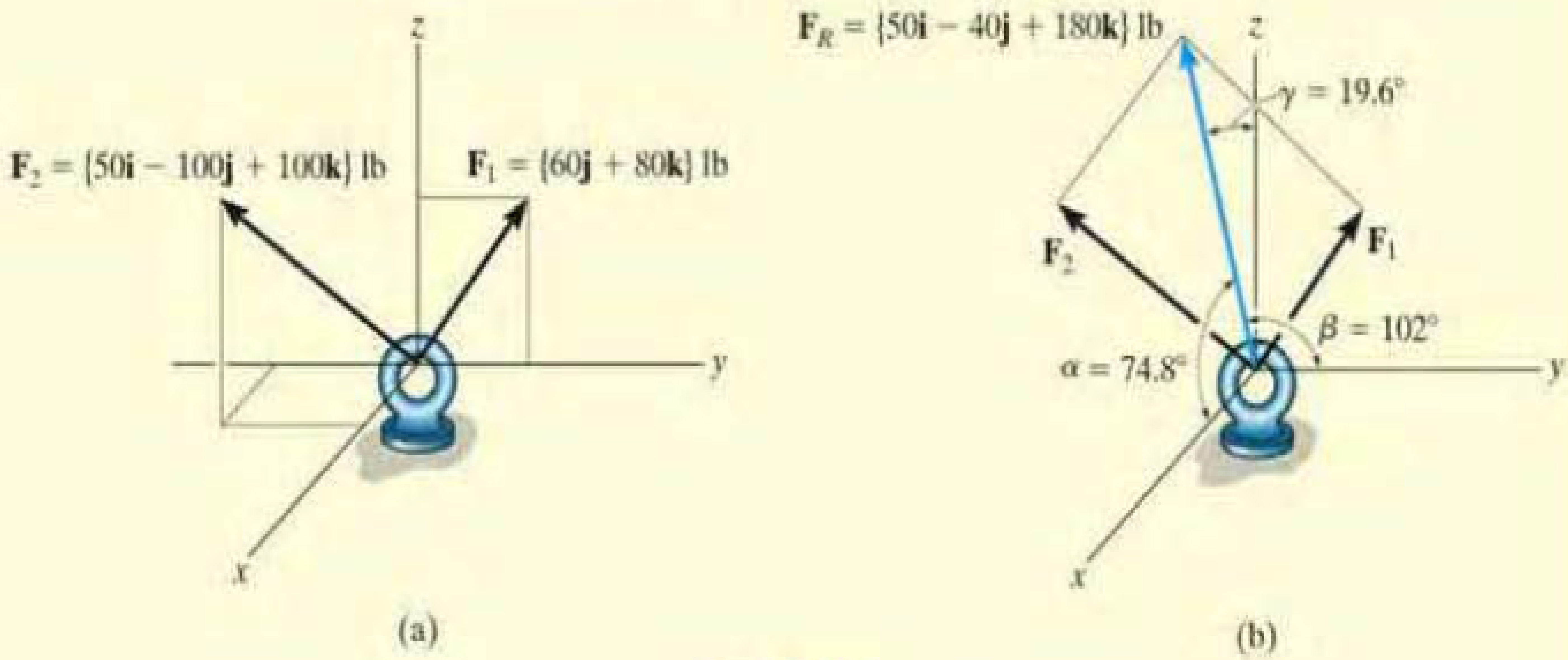


Fig. 2–31

SOLUTION

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–31*a*.

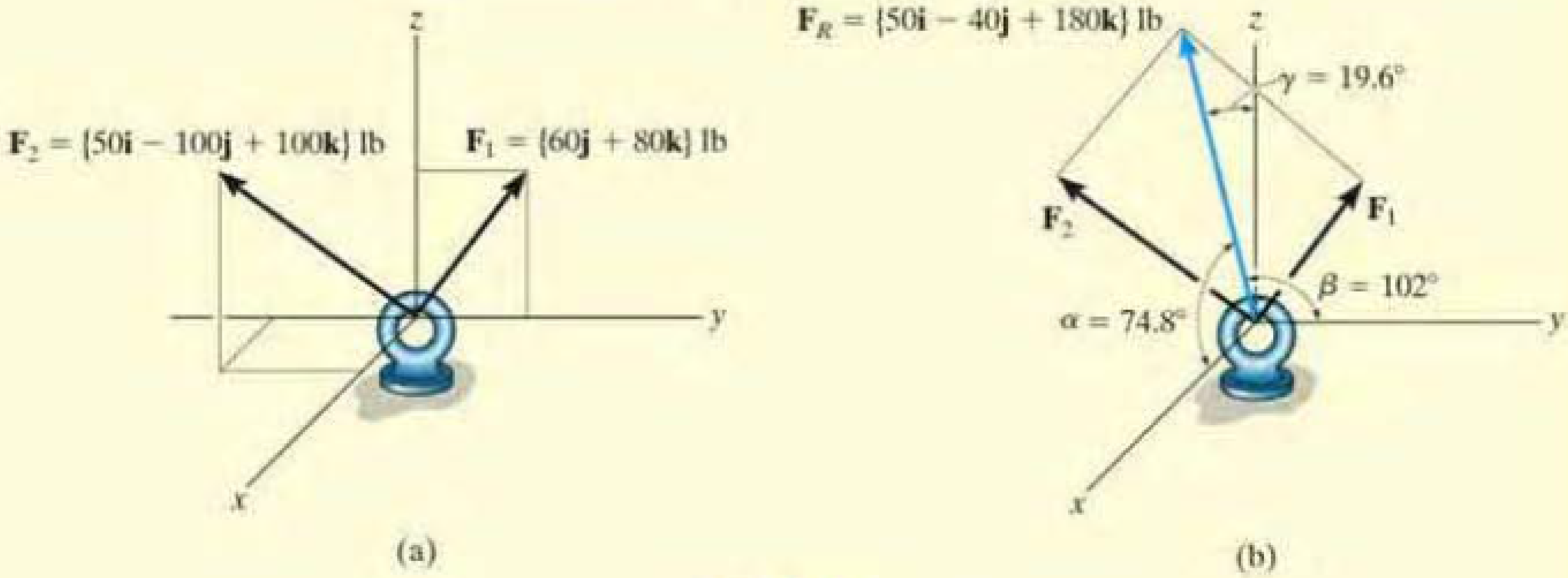


Fig. 2–31

SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in Fig. 2–31*b*, is

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\}\text{ lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{ lb}$$
$$= \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\}\text{ lb}$$

The magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(50\text{ lb})^2 + (-40\text{ lb})^2 + (180\text{ lb})^2} = 191.0\text{ lb}$$
$$= 191\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles α, β, γ are determined from the components of the unit vector acting in the direction of $\mathbf{F}_R$.

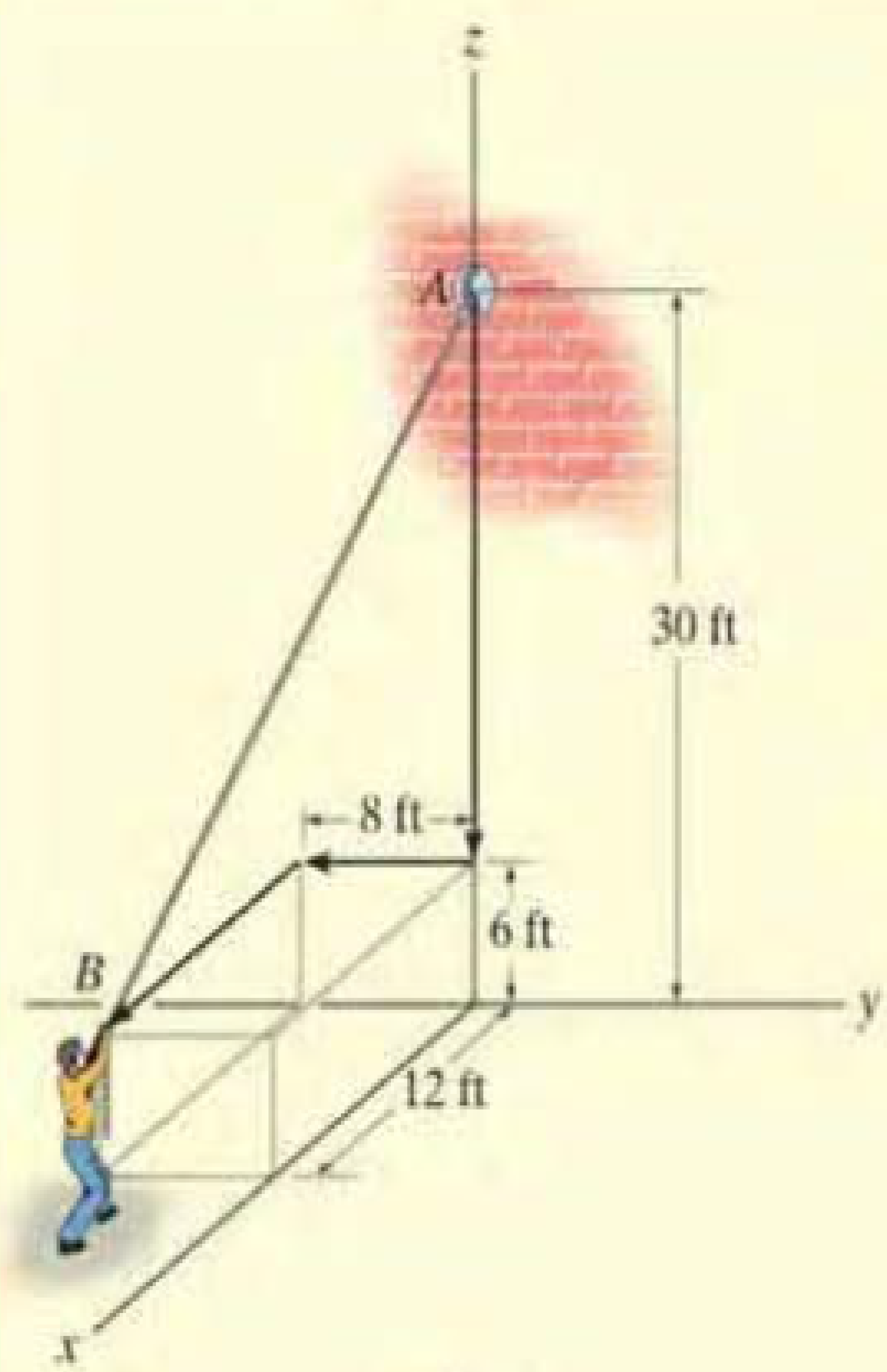
$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{50}{191.0}\mathbf{i} - \frac{40}{191.0}\mathbf{j} + \frac{180}{191.0}\mathbf{k}$$
$$= 0.2617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$$

so that

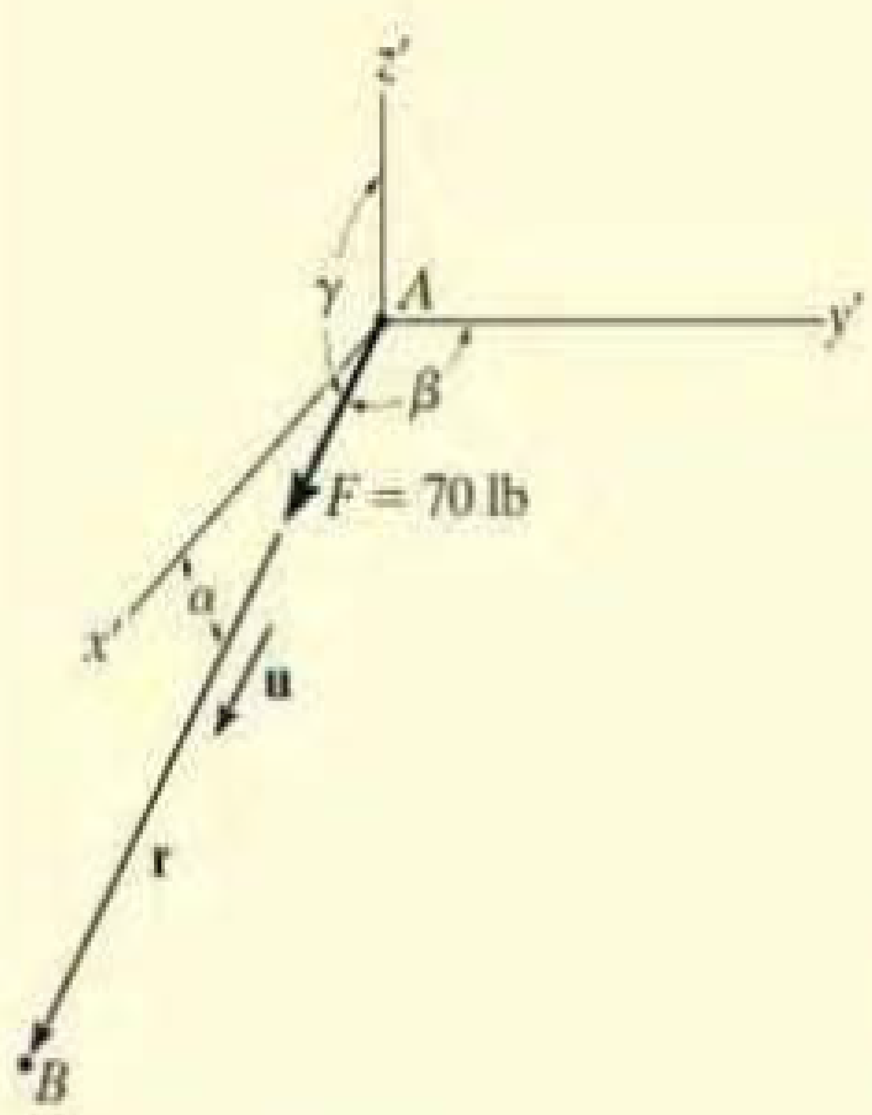
$$\cos\alpha = 0.2617 \qquad \alpha = 74.8° \qquad \textit{Ans.}$$
$$\cos\beta = -0.2094 \qquad \beta = 102° \qquad \textit{Ans.}$$
$$\cos\gamma = 0.9422 \qquad \gamma = 19.6° \qquad \textit{Ans.}$$

These angles are shown in Fig. 2–31*b*.

NOTE: In particular, notice that $\beta > 90°$ since the $\mathbf{j}$ component of $\mathbf{u}_{F_R}$ is negative. This seems reasonable considering how $\mathbf{F}_1$ and $\mathbf{F}_2$ add according to the parallelogram law.



(a)



(b)

Fig. 2–39

The man shown in Fig. 2–39*a* pulls on the cord with a force of 70 lb. Represent this force acting on the support *A* as a Cartesian vector and determine its direction.

SOLUTION

Force **F** is shown in Fig. 2–39*b*. The *direction* of this vector, **u**, is determined from the position vector **r**, which extends from *A* to *B*. Rather than using the coordinates of the end points of the cord, **r** can be determined *directly* by noting in Fig. 2–39*a* that one must travel from *A* {-24**k**} ft, then {-8**j**} ft, and finally {12**i**} ft to get to *B*. Thus,

$$\mathbf{r} = \{12\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}\} \text{ ft}$$

The magnitude of **r**, which represents the *length* of cord *AB*, is

$$r = \sqrt{(12 \text{ ft})^2 + (-8 \text{ ft})^2 + (-24 \text{ ft})^2} = 28 \text{ ft}$$

Forming the unit vector that defines the direction and sense of both **r** and **F**, we have

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}$$

Since **F**, has a *magnitude* of 70 lb and a *direction* specified by **u**, then

$$\mathbf{F} = F\mathbf{u} = 70 \text{ lb}\left(\frac{12}{28}\mathbf{i} - \frac{8}{28}\mathbf{j} - \frac{24}{28}\mathbf{k}\right)$$

$$= \{30\mathbf{i} - 20\mathbf{j} - 60\mathbf{k}\} \text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles are measured between **r** (or **F**) and the *positive axes* of a localized coordinate system with origin placed at *A*, Fig. 2–39*b*. From the components of the unit vector:

$$\alpha = \cos^{-1}\left(\frac{12}{28}\right) = 64.6^\circ \qquad \textit{Ans.}$$

$$\beta = \cos^{-1}\left(\frac{-8}{28}\right) = 107^\circ \qquad \textit{Ans.}$$

$$\gamma = \cos^{-1}\left(\frac{-24}{28}\right) = 149^\circ \qquad \textit{Ans.}$$

NOTE: These results make sense when compared with the angles identified in Fig. 2–39*b*.

The force in Fig. 2–40*a* acts on the hook. Express it as a Cartesian vector.

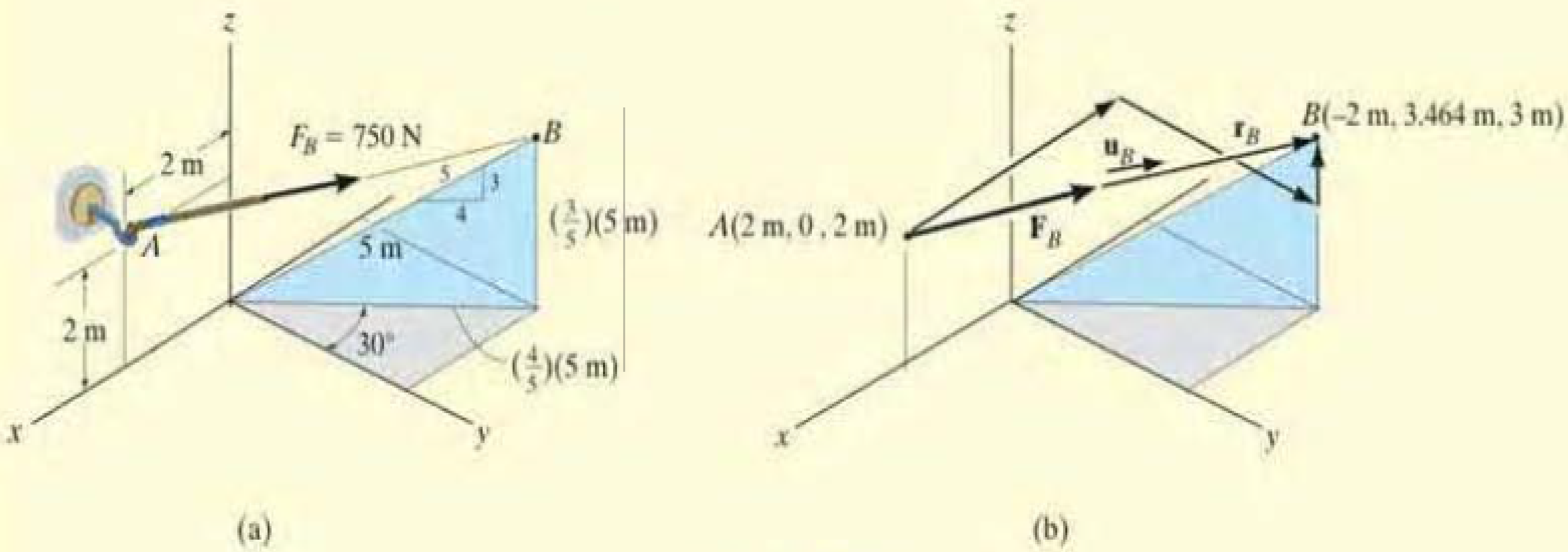


Fig. 2–40

SOLUTION

As shown in Fig. 2–40*b*, the coordinates for points *A* and *B* are

$$A(2 \text{ m}, 0, 2 \text{ m})$$

and

$$B\left[-\left(\frac{4}{5}\right)5 \sin 30^\circ \text{ m}, \left(\frac{4}{5}\right)5 \cos 30^\circ \text{ m}, \left(\frac{3}{5}\right)5 \text{ m}\right]$$

or

$$B(-2 \text{ m}, 3.464 \text{ m}, 3 \text{ m})$$

Therefore, to go from *A* to *B*, one must travel $\{4\mathbf{i}\}$ m, then $\{3.464\,\mathbf{j}\}$ m, and finally $\{1\,\mathbf{k}\}$ m. Thus,

$$\mathbf{u}_B = \left(\frac{\mathbf{r}_B}{r_B}\right) = \frac{\{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}\} \text{ m}}{\sqrt{(-4 \text{ m})^2 + (3.464 \text{ m})^2 + (1 \text{ m})^2}}$$

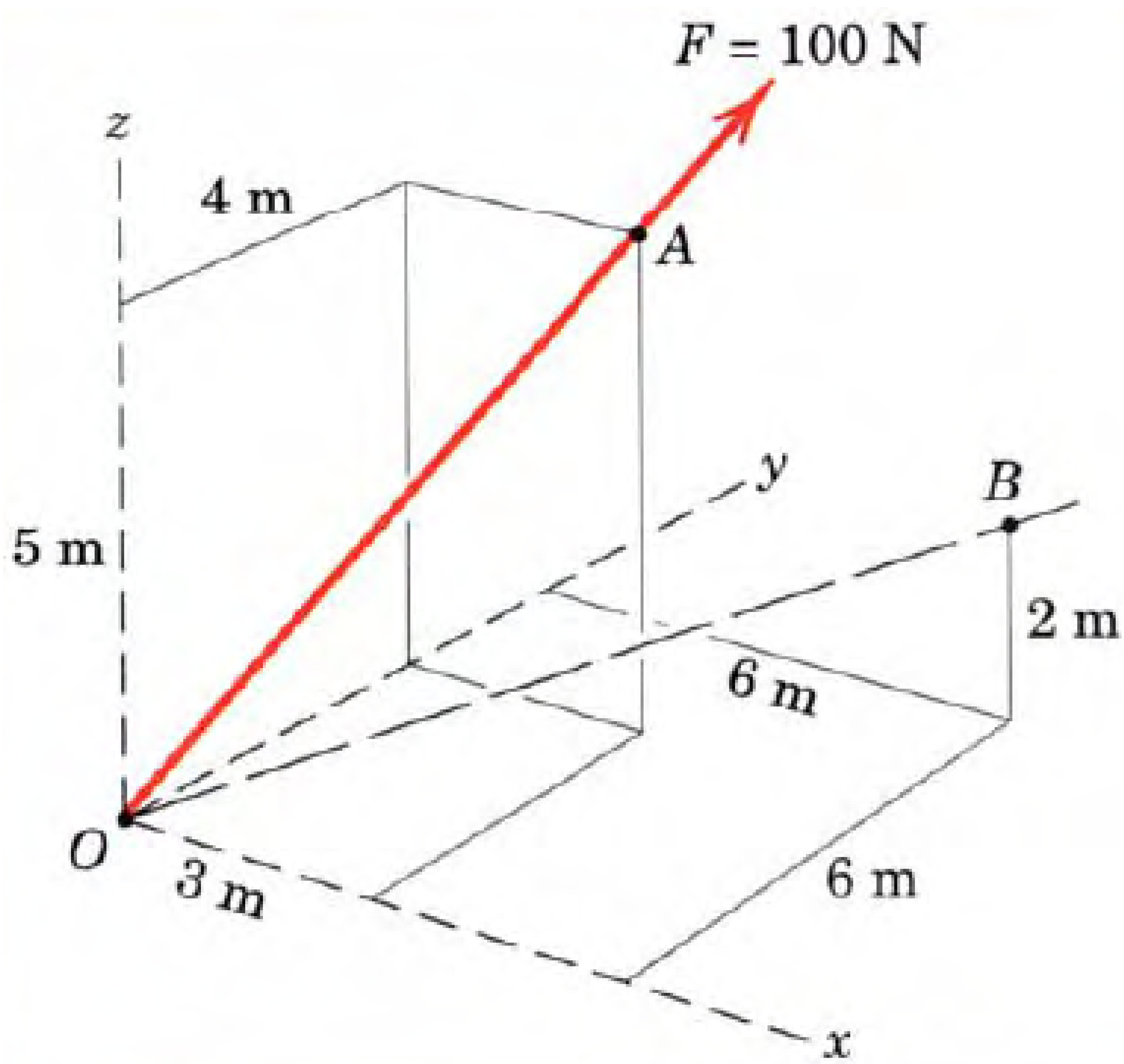
$$= -0.7428\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k}$$

Force $\mathbf{F}_B$ expressed as a Cartesian vector becomes

$$\mathbf{F}_B = F_B\mathbf{u}_B = (750 \text{ N})(-0.74281\mathbf{i} + 0.6433\mathbf{j} + 0.1857\mathbf{k})$$

$$= \{-557\mathbf{i} + 482\mathbf{j} + 139\mathbf{k}\} \text{ N} \qquad \textit{Ans.}$$

A force **F** with a magnitude of 100 N is applied at the origin O of the axes x-y-z as shown. The line of action of **F** passes through a point A whose coordinates are 3 m, 4 m, and 5 m. Determine (a) the x, y, and z scalar components of **F**, (b) the projection F_{xy} of **F** on the x-y plane, and (c) the projection F_{OB} of **F** along the line OB.



Solution.

Part (a).

$$F\frac{\overrightarrow{OA}}{\overline{OA}} = 100\left[\frac{3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}}{\sqrt{3^2 + 4^2 + 5^2}}\right]$$

$$= 100[0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}]$$
$$= 42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k}\ \text{N}$$ *Ans.*

The desired scalar components are thus

$$F_x = 42.4\ \text{N} \qquad F_y = 56.6\ \text{N} \qquad F_z = 70.7\ \text{N}$$

$$proj\left(\vec{F}/plan\ldots xy\right) = \vec{F}\cdot\frac{\overline{oA}}{|oA|} = (42.4, 56.6, 70.7)\cdot\frac{(3,4,0)}{\sqrt{3^2+5^2}} = 70.7$$

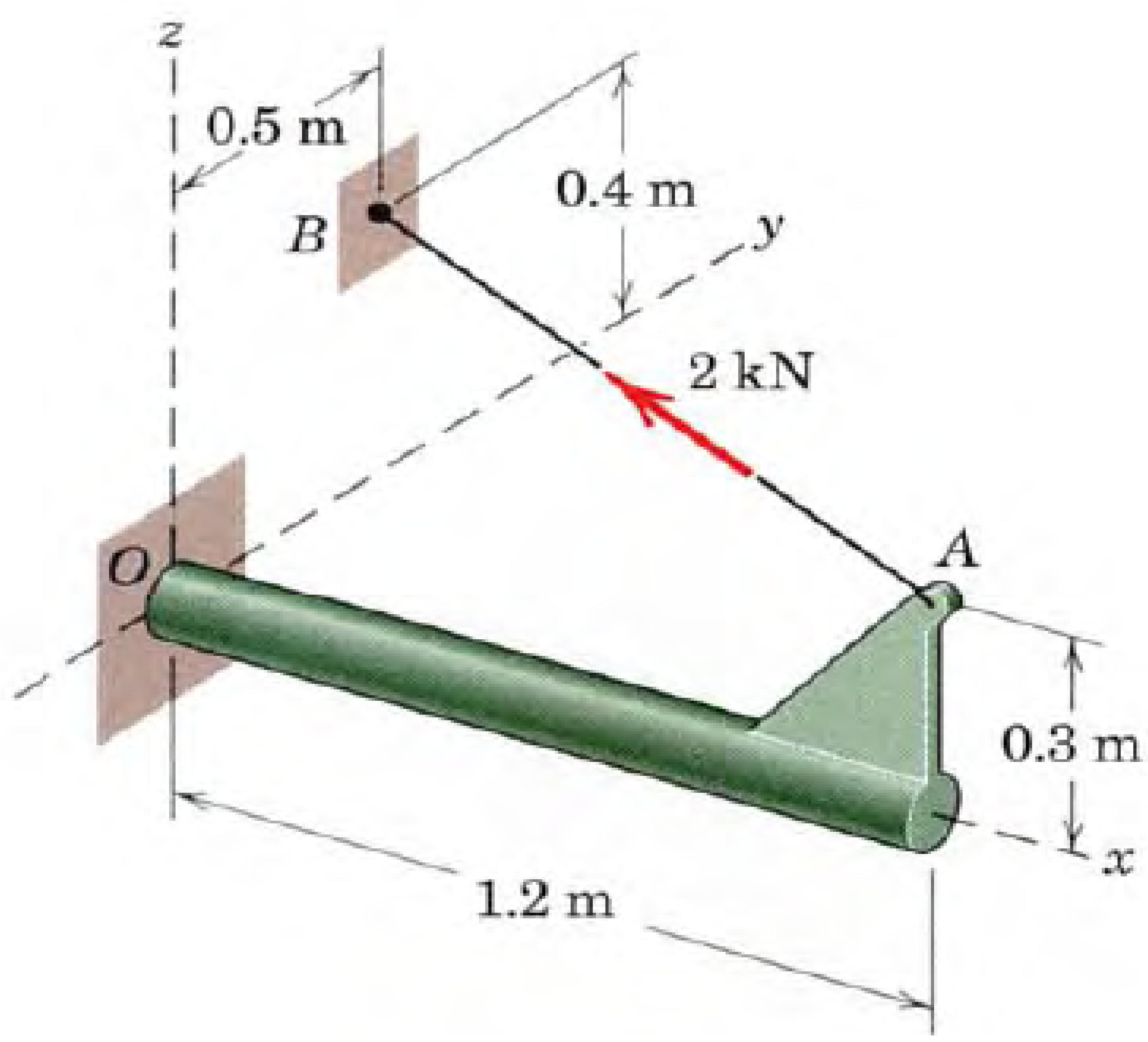
Ans.

Part (c).

$$proj\left(\vec{F}/\overline{oB}\right) = \vec{F}\cdot\frac{\overline{oB}}{|oB|} = (42.4, 56.6, 70.7)\cdot\frac{(6,6,2)}{\sqrt{6^2+6^2+2^2}} = 84.4$$ *Ans.*

Problem

The cable exerts a tension of 2 kN on the fixed bracket at A. Write the vector expression for the tension **T**.



Problem

Determine the projected component of the force

F_{AB} = 560 N acting along cable AC. Express the result as a Cartesian vector.